Lecture 18 — 03/27/2019

Electromagnetic Plane Waves (Cont'd)

Propagation of Waves in Conducting Media

For a monochromatic wave of frequency ω in a conducting medium with conductivity σ, we have:

$$\vec{\nabla}\times\vec{H} = \vec{J} + \epsilon\frac{\partial\vec{E}}{\partial t} = \sigma\vec{E} - i\omega\epsilon\vec{E} = -i\omega\underbrace{\left(\epsilon + \frac{i\sigma}{\omega}\right)}_{\epsilon_{eff}}\vec{E}$$

In a highly conducting medium, and for sufficiently low frequencies, $\frac{\sigma}{\omega} \gg |\epsilon|$, and hence:

$$\epsilon_{eff} \simeq \frac{i\sigma}{\omega} \Rightarrow n_{eff} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \simeq \sqrt{\frac{\mu\sigma}{\omega\mu_0\epsilon_0}}\,\frac{1+i}{\sqrt{2}}$$

For a non-magnetic medium, $\mu = \mu_0$, we have:

$$n_{eff} \simeq \sqrt{\frac{\sigma}{\omega\epsilon_0}}\,\frac{1+i}{\sqrt{2}}$$

For a wave propagating in the z direction:

$$\vec{E} = \vec{E}_0\, e^{i(kz-\omega t)}, \quad k^2 = \frac{\omega^2}{c^2} n_{eff}^2$$

Thus:

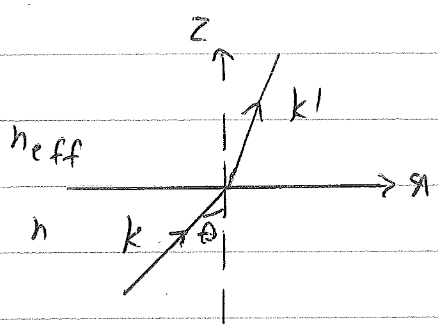
$$k = \frac{\omega}{c}\sqrt{\frac{\sigma}{2\omega\epsilon_0}}\,(1+i) \Rightarrow \vec{E} = \vec{E}_0\, e^{-\frac{\omega}{c}\sqrt{\frac{\sigma}{2\omega\epsilon_0}}\, z}\, e^{i\left(\frac{\omega}{c}\sqrt{\frac{\sigma}{2\omega\epsilon_0}}\, z - \omega t\right)}$$

$$= \vec{E}_0\, e^{-\frac{z}{\delta}}\, e^{i\left(\frac{z}{\delta} - \omega t\right)}, \quad \delta \equiv c\left(\frac{2\epsilon_0}{\omega\sigma}\right)^{\frac{1}{2}} \text{ ; skin depth}$$

We can now use all of the formulas for reflection and transmission coefficients by replacing n' with $n_{eff} \simeq \sqrt{\frac{\sigma}{\omega\epsilon_0}}\,\frac{1+i}{\sqrt{2}}$. For example, consider the interface between a medium with the index of refraction n and a highly conducting medium:

$$\underset{\uparrow}{k'_x = k_x} = \frac{\omega}{c}\, n\sin\theta, \quad k'^2 = \frac{\omega^2}{c^2}\, n_{eff}^2$$

Snell's law



$$k'_z = \sqrt{k'^2 - k'^2_x} = \frac{\omega}{c}\sqrt{\frac{i\sigma}{\omega\epsilon_0} - n^2\sin^2\theta} \underset{\uparrow}{\simeq} \frac{1}{\delta}(1+i)$$

for a highly conducting medium

Hence:

$$\vec{E}' = \vec{E}'_0\, e^{-\frac{z}{\delta}}\, e^{i\left(k'_x x + \frac{1}{\delta} z\right) - i\omega t}, \quad \vec{k}' = \left(k'_x, 0, \frac{1+i}{\delta}\right) \simeq \left(0, 0, \frac{1+i}{\delta}\right)$$

This implies that $\vec{k}'$ is essentially normal to the interface in this case.

The amplitude reflection coefficient is given by:

$$\boxed{r = \frac{n\cos\theta - \sqrt{n_{eff}^2 - n^2\sin^2\theta}}{n\cos\theta + \sqrt{n_{eff}^2 - n^2\sin^2\theta}} \simeq -1}$$

Therefore, good conductors are also good reflectors. The amplitude transmission coefficient is:

$$t = \frac{2n\cos\theta}{n\cos\theta + \sqrt{n_{eff}^2 - n^2\sin^2\theta}} \simeq \frac{2n\cos\theta}{n_{eff}} = \boxed{n\sqrt{\frac{2\omega\epsilon_0}{\sigma}}\,(1-i)\cos\theta}$$

The transmitted wave thus has a phase shift of $-\frac{\pi}{4}$ relative to the incident wave. Considering normal incidence, $\theta=0$, we find:

$$E_{trans} = tE_0 \Rightarrow |E_{trans}| \ll E_0$$

$$\vec{H}_{trans} = \frac{1}{\mu_0}\frac{\vec{k}' \times \vec{E}_{trans}}{\omega} \simeq \frac{1}{\mu_0\omega}\frac{1+i}{\delta}\hat{z}\times\vec{E}_{trans} \Rightarrow |\vec{H}_{trans}| = \frac{2n}{c}\frac{E_0}{\mu_0}$$

This is consistent with the fact that the tangential component of $\vec{H}_{trans}$ is continuous across the interface (note that $|\vec{H}_{inc}|_{0^-} = |\vec{H}_{ref}|_{0^-} = \frac{n}{c}\frac{E_0}{\mu_0}$).

A Microscopic Theory of Linear Dielectric/Conductor Responce

The linear response of a dielectric material to an electric field is given in terms of its dielectric constant. A simple model assumes that this response is due to bound electrons regarded as isotropic harmonic oscillators in their vibratory motion about the nuclei to which they are bound. Quantum mechanically, this is justified as long as the atoms are weakly excited and there is little transfer to the higher excited levels.

We can therefore treat the linear responce problem as that of a charged harmonic oscillator with characteristic frequency ω_0 interacting with an electromagnetic field of frequency ω. Then:

$$m_e \ddot{\vec{x}} = \underbrace{-m_e\omega_0^2 \vec{x}}_{\text{restoring force}} \underbrace{- m_e \gamma \dot{\vec{x}}}_{\text{damping force}} \underbrace{- e\vec{E}}_{\text{external force}}$$

The magnetic Lorentz force can be neglected since for a plane wave $|\vec{F}_{mag}| \sim \frac{v}{c} |\vec{F}_{elec}| << |\vec{F}_{elec}|$. Then:

$$\ddot{\vec{x}} + \gamma \dot{\vec{x}} + \omega_0^2 \vec{x} = -\frac{e}{m_e} \vec{E}(\vec{x}, t)$$

Since $\vec{E}_{(\vec{x},t)} \propto e^{-i\omega t}$, the steady-state solution of $\vec{x}$ has the same time dependence, resulting in:

$$\vec{x}(-\omega^2 - i\gamma\omega + \omega_0^2) = \frac{-e}{m_e}\vec{E}e^{-i\omega t} \Rightarrow \vec{x} = \frac{-e}{m_e(\omega_0^2 - \omega^2 - i\gamma\omega)}\vec{E}$$

The dipole moment density of the medium is given by:

$$\vec{P} = \sum_j -n_j e \vec{x}_j$$

Here n_j is the number density of electrons with resonance frequency ω_j, which is related to the number density of atoms n through $n_j = f_j n$ (where $\sum_j f_j = Z$). Hence:

$$\vec{P} = \frac{ne^2}{m_e}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\vec{E} \Rightarrow \boxed{\chi(\omega) = \frac{ne^2}{m_e\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}}$$

Here, $\chi(\omega)$ is the susceptibity of the medium at frequency ω, which leads to:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \chi(\omega)$$

$$\boxed{\mathrm{Re}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) = 1 + \frac{ne^2}{m_e\varepsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}, \quad \mathrm{Im}\left(\frac{\varepsilon(\omega)}{\varepsilon}\right) = \frac{ne^2}{m_e\varepsilon_0}\sum_j \frac{f_j\gamma_j\omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}}$$

There fore:

$$n(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} = \sqrt{Re\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) + i\, Im\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right)}$$

$\Downarrow$ assuming $\mu = \mu_0$

For weak absorption, we have:

$$n(\omega) \simeq \left[Re\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right)\right]^{1/2} + \frac{i}{2}\,\frac{Im\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right)}{\left[Re\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right)\right]^{1/2}}$$

The intensity of an electromagnetic wave in such a medium is attenuated according to:

$$\exp\left[-\frac{\omega}{c}\,\frac{Im\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right)}{\left[Re\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right)\right]^{1/2}}\, z\right] \qquad \text{(Beer's law)}$$

At low frequencies, there is a number of free electrons (Corresponding to $\omega_j = 0$) and those for which $\omega \ll \omega_j$. In this limit, we have:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 + \frac{Ne^2}{m_e\varepsilon_0}\sum_{j\neq 0}\frac{f_j}{\omega_j^2} - \frac{Ne^2}{m_e\varepsilon_0}\,\frac{f_0}{\omega^2 + i\gamma_0\omega} = \frac{\varepsilon_b}{\varepsilon_0} + \frac{i\,\sigma(\omega)}{\omega\varepsilon_0}$$

Here $\frac{\varepsilon_b}{\varepsilon_0}$ is essentially independent from ω, and:

$$\boxed{\sigma(\omega) = \frac{Ne^2 f_0}{m_e(\gamma_0 - i\omega)} = \frac{\sigma_0}{1 - \frac{i\omega}{\gamma_0}}, \quad \sigma_0 \equiv \frac{Ne^2 f_0}{m\gamma_0}}$$

"Drude model of Conductivity"

In the high-frequency limit, $\omega \gg \omega_j$ for all j, resulting in:

$$\frac{\varepsilon(\omega)}{\varepsilon} \simeq 1 + \frac{n e^2}{m_e \varepsilon_0} \underbrace{\sum_j f_j}_{Z} \frac{1}{-\omega^2} = 1 - \frac{n_0 e^2}{m_e \varepsilon_0 \omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \qquad (n_0 = nZ)$$

Where:

$$\omega_p^2 \equiv \frac{n_0 e^2}{m_e \varepsilon_0} \qquad (\omega_p : \text{"plasma frequency"})$$

It is seen that $\frac{\varepsilon(\omega)}{\varepsilon} < 0$ for $\omega < \omega_p$. This gives rise to an imaginary index of refraction at these frequencies:

$$n^2_{(\omega)} = \frac{\varepsilon(\omega)}{\varepsilon_0} < 0 \Rightarrow \frac{c^2 k^2}{\omega^2} < 0 \Rightarrow k^2 < 0 \Rightarrow k = i|k|$$

This implies no propagation in the medium at frequencies below the plasma frequency. On the other hand, for $\omega > \omega_p$, the wave propagates in the medium as $n(\omega)$ is real in this case.

Plasmas in a Static Magnetic Field

Consider a static magnetic field in the same direction as the wave is propagating (say the z direction). In this case, we have:

$$m_e \ddot{\vec{X}} = -e\vec{E}e^{-i\omega t} - e\dot{\vec{X}} \times \vec{B}_0 \qquad (\vec{B}_0 = B_0\hat{z})$$

We note that $|\vec{B}_0| \gg |\vec{E}|$ is needed in order for the magnetic field to have a significant effect. For the circular polarization states of the wave, we have:

$$\vec{E}_\pm = E_\pm \hat{e}_\pm \quad , \quad \hat{e}_\pm \equiv \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}}$$

The steady-state solutions for $\vec{x}$ in the presence of the circularly-polarized wave and the static magnetic field can be written as:

$$\vec{X}_\pm = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}) X_\pm e^{-i\omega t}$$

Then:

$$\dot{\vec{X}}_\pm = -i\omega \vec{X}_\pm \,,\; \ddot{\vec{X}}_\pm = -\omega^2 \vec{X}_\pm \,,\; \dot{\vec{X}}_\pm \times \vec{B}_0 = \pm \omega B_0 \vec{X}_\pm$$

Therefore:

$$X_\pm = \frac{eE_\pm}{(m_e\omega^2 \mp e\omega B_0)} \Rightarrow \vec{P}_\pm = \varepsilon_0 \chi_\pm E_\pm \hat{e}_\pm$$

This leads to:

$$\boxed{\frac{\varepsilon_\pm}{\varepsilon_0} = 1 + \chi_\pm = 1 - \frac{n_0 e^2}{(m_e\omega^2 \mp e\omega B_0)\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)}} \quad \left(\omega_B \equiv \frac{eB_0}{m_e}\right)$$

We see that the refraction indices for the two circularly-polarized states $n_{\pm} = \sqrt{\frac{\epsilon_{\pm}}{\epsilon_0}}$ are different, called "birefringence". This results in the rotation of linear polarization for $\omega > \omega_p$. To see this, let us write the electric field vector as follows:

$$\vec{E}_{(z=0)} = E_0 e^{-i\omega t} \hat{x} = \frac{1}{\sqrt{2}} (\hat{e}_+ + \hat{e}_-) E_0 e^{-i\omega t} \Rightarrow \vec{E}_{(z)} = \frac{1}{\sqrt{2}} (E_0 e^{ik_+ z} \hat{e}_+$$
$$+ E_0 e^{-ik_- z} \hat{e}_-) e^{-i\omega t}, \quad k_{\pm} = \frac{\omega}{c} n_{\pm}$$

Thus:

$$\vec{E}_{(z)} = \frac{E_0}{\sqrt{2}} e^{i(\frac{k_+ + k_-}{2})z} \Big[\underbrace{e^{\frac{i\Delta k z}{2}} \hat{e}_+}_{\hat{e}'_+} + \underbrace{e^{\frac{-i\Delta k z}{2}} \hat{e}_-}_{\hat{e}'_-} \Big] \quad \left(\Delta k = \frac{\omega}{c}(n_+ - n_-)\right)$$

This amounts to a linearly-polarized wave in the x' direction, where:

$$\hat{x}' = \frac{\hat{e}'_+ + \hat{e}'_-}{\sqrt{2}}$$

This new direction makes an angle $\frac{-\Delta k z}{2}$ with the x direction.

This phenomenon is called the "Faraday rotation".